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Design of an Effective Controller via Disturbance Accommodating Left Eigenstructure Assignment

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An effective and disturbance suppressible controller can be obtained by assigning the left eigenstructure (eigenvalues/left eigenvectors) of a system. To design such a controller, both the controllability and disturbance suppressibility should be considered simultaneously. The controllability of the system may be degraded if the left eigenstructure is chosen only to suppress the disturbance and vice versa. In this paper, a modal disturbance suppressibility measure is proposed that indicates the degree of the system's disturbance suppression performance, and a simple general left eigenstructure assignment scheme, considering both the proposed modal disturbance suppressibility measure and an improved version of a modal controllability measure, is suggested. The biorthogonality condition between modal matrices is utilized to develop the scheme. The proposed left eigenstructure assignment scheme makes it possible to achieve the desired left eigenstructure exactly if the desired left eigenvectors reside in the achievable subspace. In case the desired left eigenvectors do not reside in the achievable subspace, the left eigenvectors are assigned to the best possible set of eigenvectors in the least square sense, guaranteeing the desired eigenvalues to be achieved exactly. The proposed scheme is applied to a lateral flight control system design of an L-1011 aircraft model with wind disturbances.

I. Introduction

THE problem of eigenstructure assignment via linear state feedback control in a linear multivariable system is of vital importance in control theory and applications. The specified effect of the controller is achieved by assigning a certain set of eigenvalues and an associated set of eigenvectors to the closed-loop system. In general terms, the speed of response is determined by the assigned eigenvalues, whereas the shape of the response is furnished by the assigned eigenvectors.[1] It is well known that, apart from the case of single-input systems, specification of closed-loop eigenvalues does not uniquely determine a closed-loop system. The source of nonuniqueness can be identified as that coming from the freedom offered by state feedback beyond eigenvalue assignment, in selecting the associated eigenvectors from an admissible class.[2] The right eigenstructure (eigenvalues/right eigenvectors) is widely used to solve mode decoupling problems, whereas transient responses of a linear system having undesired disturbances are dominantly governed by a system's left eigenstructure (eigenvalues/left eigenvectors).

Andry et al.[3] and Kang and Lee[4] used right eigenstructure assignment for flight control system design, using output feedback, to decouple the modes of an aircraft. Innocenti and Stanziola[5] analyzed the performance-robustness properties of right eigenstructure assignment against the standard LQR to define a loop transfer recovery procedure similar to that of the LQG/LTR and examined the sensitivity properties of LQR and eigenstructure assignment in their application. Sobel and Cloutier[6] utilized a right eigenstructure for mode decoupling of missile dynamics. The authors in Refs. 2, 7, 8 also dealt with the problem of right eigenstructure assignment.

Zhang et al.[9] used a left eigenstructure to suppress undesired inputs, through orthogonalizing left eigenvectors to disturbance input matrix of the system of uniform flexible beam vibration control problem. Kim and Junkins[10] utilized the left eigenstructure to improve the controllability of a flexible structure system through placing actuators at optimal locations. However, Zhang et al. did not take into account the control problems, and Kim and Junkins did not consider the disturbance suppression problems.

One of the drawbacks of direct eigenstructure assignment techniques as compared with some other multivariable techniques, especially the LQG/LTR approach, is that the synthesis procedure does not guarantee stability robustness with respect to parameter variations in the plant dynamics. Sobel et al.[11] proposed a sufficient condition for the robust stability of a linear time-invariant system subject to time varying structured state space uncertainty. Using the results, Yu and Sobel[12] proposed a robust eigenstructure assignment design method that optimizes either the sufficient condition for stability or performance robustness while constraining the dominant eigenvalues to lie within chosen performance regions in the complex plane. Patton et al.[13] proposed a parameter insensitive design method using eigenstructure assignment and the method of inequalities. The problems of sensitivity to parameter variations and frequency domain analysis for characterizing robustness properties are not covered here. The objective of this paper is to develop a new left eigenstructure assignment technique for control synthesis. Thus, the focus of this paper is on the left eigenstructure assignment technique.

In control system design problems, both the controllability and the disturbance suppressibility should be considered simultaneously. Otherwise, the controllability of the system could be degraded if the left eigenstructure is chosen only to suppress the disturbance and vice versa. In the left eigenstructure assignment problem, the number of assignable left eigenvectors satisfying the prescribed design specifications are severely restricted by the ranks of the input and output matrices along with imposed conditions if output feedback is used.[14] For the existing state feedback scheme, the feedback gain matrix for left eigenstructure assignment has been given in the least square sense. Thus, the closed-loop eigenvalues as well as the associated eigenvectors may not coincide with the desired ones.

On the other hand, the problem of observer design is dual to that of the controller design. Thus, the assignment of the right eigenstructure of the observer corresponds to the assignment of the left eigenstructure of the dual controller. Patton and Willcox[15] and Patton et al.[16] first demonstrated the left eigenstructure assignment

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approach (the assignment technique is the same as that of the existing right eigenstructure[3] of the dual controller) to robust fault detection. By assigning the left eigenstructure of a closed-loop system such as an observer, a well-defined residual signal can be completely decoupled from the disturbances.[17] In Refs. 17–20, Patton et al. used a left eigenstructure for optimum disturbance decoupling or for minimizing the effects of initial condition disturbances. Especially, Patton and Chen[17] proposed another new method for the assignment of the right eigenstructure of the observer (left eigenstructure of the dual controller) in the least square sense, which may cause a discrepancy between the closed-loop eigenvalues and the desired eigenvalues.

In this paper, a simple and general left eigenstructure assignment scheme, based upon the biorthogonality condition between the right and left modal matrices of a system, is proposed. The proposed scheme enables designers to consider both the modal disturbance suppressibility and modal controllability in designing control systems. It also guarantees that the closed-loop eigenvalues are exactly achieved if the system is controllable, and the desired left eigenvectors are also achieved exactly if the eigenvectors are achievable. In case the desired left eigenvectors do not lie in the achievable subspace, the eigenvalues are exactly achieved and the left eigenvectors are assigned to the best possible set of eigenvectors in the least square sense. For the suggested left eigenstructure assignment scheme, a modal disturbance suppressibility measure is proposed with a new version of Hamdan and Nayfeh's[21] measure of modal controllability that reflects the magnitude of each element of a control input matrix, guaranteeing consistency with their gross measure of controllability.

Consider a linear time invariant multivariable controllable system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ef(t) \tag{1}$$

$$u(t) = Kx(t) \tag{2}$$

where $x \in R^N$, $u \in R^m$, and $f \in R^n$ denote the state, control, and disturbance vectors, respectively; A, B, E, and K are real constant matrices of appropriate dimensions; and rank $B = m \neq 0$. The response of the given system due to control input $u(t)$ and disturbance $f(t)$ with zero initial conditions is represented using modal matrices by[22]

$$x(t) = \Phi \int_0^t e^{\Lambda(t-\tau)}\{\Psi^T Bu(\tau) + \Psi^T Ef(\tau)\}\,\mathrm{d}\tau \tag{3}$$

where Λ is the diagonal matrix of desired eigenvalues, and Φ and Ψ denote the right and left modal matrices of the given system, respectively.

Note, from Eq. (3), that the response to disturbances can be eliminated if the columns of Ψ are orthogonal to the columns of the disturbance input matrix E. Note also that the control efforts are effectively transferred (that is, the desired maneuver is achieved with smaller control efforts) if the left eigenvectors are parallel to the columns of the control input matrix B. Therefore, for both effective control and disturbance suppression, it is desired that the left eigenvectors of the system are simultaneously orthogonal to the columns of E and parallel to the columns of B. Then, the corresponding system can be manipulated with small control efforts without being disturbed by the disturbance input. However, we can expect cases in which some columns of B are parallel to those of E, in which case it is impossible to assign left eigenvectors to be simultaneously parallel to the columns of B and orthogonal to those of E. In these cases, a tradeoff should be made between the controllability and the disturbance suppressibility.

Recently, several concepts of measure of controllability,[10,21–24] which represent qualitative degree of controllability, are developed and are applied to find the optimal locations of actuators. The measures are basically sensitivity measures for control inputs.[25] Hamdan and Nayfeh[21] proposed a modal controllability measure from angles between the left eigenvectors of the system matrix A and the columns of the control input matrix B for the system described by the triple (A, B, C). The results of Hamdan and Nayfeh are extended by Kim and Junkins[10] by introducing a new controllability index that combines Hamdan and Nayfeh's ideas with Skelton's[26] modal cost analysis. However, the measure of modal controllability proposed by Hamdan and Nayfeh does not reflect the magnitude of each element of B. The higher norm of a column of the matrix B indicates that more power is injected with the same input and thus yields better controllability. Thus, we improve the measure of modal controllability proposed by Hamdan and Nayfeh to reflect the magnitude of each element of the control input matrix B, guaranteeing consistency with their gross measure of controllability.

Similarly, for a linear system with disturbances, a quantitative measure of modal suppressibility is also required to investigate how suppressible the system's undesired disturbances are. We propose a modal disturbance suppressibility measure by using generalized angles between the left eigenvectors of the system matrix A and the columns of the disturbance input matrix E of the system. The proposed disturbance suppressibility measure coincides with the intuitive result used by Zhang et al.,[9] where the left eigenvectors were tried to be placed orthogonal to the columns of the matrix E to minimize the disturbance effect on the system.

The proposed design scheme is applied to the lateral dynamics of the L-1011 aircraft at a cruise flight condition with the existence of atmospheric winds without considering parameter variations.

II. Problem of the Existing Eigenstructure Assignment with State Feedback

Consider Eq. (1) in Sec. I. If a constant real state feedback [Eq. (2)] is applied to Eq. (1), the closed-loop system becomes

$$\dot{x}(t) = (A + BK)x(t) + Ef(t) \tag{4}$$

Let $\Lambda = \{\lambda_1, \ldots, \lambda_s\}$ be a self-conjugate set of complex numbers and let $\{d_i \mid i = 1, \ldots, s;\, s \leq N\}$ be a set of positive integers satisfying $\sum_{i=1}^{s} d_i = N$. It is shown[27] that if the closed-loop system has s blocks of order $d_1, \ldots, d_s$, in its Jordan canonical form, then there are s corresponding generalized right and left eigenvector chains defined by

$$\begin{aligned} (A + BK - \lambda_i I_N)\phi_{i1} &= 0 \\ (A + BK - \lambda_i I_N)\phi_{ij} &= \phi_{ij-1}, \qquad j = 2, \ldots, d_i \end{aligned} \tag{5}$$

$$\begin{aligned} \psi_{id_i}^{\mathrm{T}}(A + BK - \lambda_i I_N) &= 0, \\ \psi_{ij}^{\mathrm{T}}(A + BK - \lambda_i I_N) &= \psi_{ij+1}^{\mathrm{T}}, \qquad j = 1, \ldots, d_i - 1 \end{aligned} \tag{6}$$

where ϕ_{ij} and ψ_{ij} are the generalized right and left eigenvectors of the given system, respectively. The problem of eigenstructure assignment is then to choose the feedback gain matrix K such that the required conditions for the eigenvalues and eigenvectors are satisfied and therefore may be considered as an inverse eigenvalue problem.

The right modal matrix Φ can be decoupled by

$$\Phi = [\Phi_1, \Phi_2, \ldots, \Phi_i, \ldots, \Phi_s], \qquad \Phi_i = [\phi_{i1}, \phi_{i2}, \ldots, \phi_{id_i}]_{N \times d_i}$$

In the following, matrices Ψ and W are defined similarly, and the superscript $(\cdot)^*$ denotes the conjugate of a given complex vector or scalar $(\cdot)$.

The following theorem gives necessary and sufficient conditions for the existence of K, which yields prescribed eigenvalues and eigenvectors.

Theorem 2.1[14]: There exists a real matrix K satisfying Eq. (5) if and only if the following conditions are satisfied.

1) The vectors in $\{\phi_{ij} \mid i = 1, \ldots, s;\, j = 1, \ldots, d_i\}$ are linearly independent in C^N and $\lambda_i = \lambda_k^*$ implies $\phi_{ij} = \phi_{kj}^*$.

2) There exists a set of vectors $\{w_{ij} \mid i = 1, \ldots, s;\, j = 1, \ldots, d_i\}$ such that

$$[A - \lambda_i I_N \mid B]\begin{bmatrix} \phi_{ij} \\ -- \\ w_{ij} \end{bmatrix} = \phi_{ij-1}, \qquad \phi_{i0} = 0$$

The preceding theorem indicates that closed-loop eigenstructure assignment by state feedback is constrained by the requirement that the generalized right eigenvectors should lie in certain subspaces.

The contents in Theorem 2.1 are useful in the controller design using eigenstructure assignment. However, the theorem provides only a right eigenstructure assignment scheme.

Now, we shall point out the emerging drawback when we try to assign a left eigenstructure of a system using Theorem 2.1. Consider the case with distinct eigenvalues [that is, $d_i=1$, for all $i = 1, \ldots, N$ in Eq. (6)] for simplicity. Then, the matrix form of Eq. (6) can be represented by

$$\left[\lambda_i I_N - A^T | I_N\right] \begin{bmatrix} \psi_{i1} \\ ----- \\ -K^T B^T \psi_{i1} \end{bmatrix} = 0 \tag{7}$$

One can find that the feedback gain matrix K could be given in the least square sense in Eq. (7), since $m \leq N$ in general. Therefore, it is expected that the achieved closed-loop eigenvalues may not coincide with the desired eigenvalues. Hence, the left eigenstructure assignment scheme by state feedback based only upon Theorem 2.1 is of little use. This fact has motivated our work, and thus a novel left eigenstructure assignment scheme by state feedback, which removes the drawback, is proposed in this paper.

III. Measures of Controllability and Disturbance Suppressibility

The measure of controllability is important because it says how easily the system can be manipulated with small energy. Similarly, the measure of disturbance suppressibility is important because it says how much the disturbance affects the system performance. In the following subsection, the improved version of Hamdan and Nayfeh's[21] modal controllability measure is proposed. After that, a measure of modal disturbance suppressibility is proposed to investigate how suppressible the system's undesired disturbances are. Proposed measures of modal controllability and disturbance suppressibility will be used as a design criterion to assign the left eigenstructure of a system in Sec. IV.

Measures of Modal and Gross Controllability

Viswanathan and Longman,[23] Viswanathan et al.,[24] and Klein et al.[28] have introduced several approaches to measure the degree of controllability of a linear dynamical system and then developed numerical methods to generate approximate values of their controllability measures for any linear time-invariant system. We propose a varied version of the measure of modal controllability suggested by Hamdan and Nayfeh to reflect the magnitude of each element of a control input matrix B.

Definition 3.1: A scalar measure of modal controllability μ_{ij} of the ith mode from the jth actuator input of the given system is defined for $i = 1, \ldots, N$; $j = 1, \ldots, m$, as follows:

$$\mu_{ij} = (\cos\theta_{ij})\|b_j\|_2 = \frac{|\psi_i^T b_j|}{\|\psi_i\|_2} \tag{8}$$

Remarks: The improved measure of modal controllability reflects the magnitude of each element of B. When $\|b_j\| = 1 (j = 1, \ldots, m)$, as a special case, the improved measure of modal controllability coincides with Hamdan and Nayfeh's result.

Definition 3.2[21]*:* A gross measure of controllability ρ_i of the ith mode from all inputs is defined as follows:

$$\rho_i = \|g_i\|_2, \qquad g_i = \frac{\psi_i^T B}{\|\psi_i\|_2} \tag{9}$$

Measures of Modal and Gross Disturbance Suppressibility

In this subsection, we propose a modal disturbance suppressibility measure and a gross disturbance suppressibility measure for a linear system having undesired disturbances to deal with the disturbance suppression problems.

Definition 3.3: A measure of modal disturbance suppressibility ν_{ik} of the ith mode from the kth disturbance input of the given system is defined for $i = 1, \ldots, N$; $k = 1, \ldots, n$ as follows:

$$\nu_{ik} = (\cos\gamma_{ik})\|e_k\|_2 = \frac{|\psi_i^T e_k|}{\|\psi_i\|_2} \tag{10}$$

where γ_{ik} is the angle between the kth column vector e_k of the disturbance input matrix E and the ith left eigenvector ψ_i of the left modal matrix Ψ of the given system.

Remarks: Definition 3.3 assigns a measure of disturbance suppressibility for the ith mode from the kth disturbance input, which is proportional to the magnitude of e_k as well as the angle between the subspaces spanned by e_k and ψ_i. When they are orthogonal, the measure of disturbance suppressibility is mapped to zero. This means that the kth disturbance is completely suppressed and does not appear in the ith mode. Note that a smaller value of the disturbance suppressibility measure represents better suppression performance of the disturbance.

Let us define the following $(n \times n)$ diagonal matrix V:

$$V = \mathrm{diag}[\|e_1\|_2, \|e_2\|_2, \ldots, \|e_n\|_2]$$

and let H, the matrix that is composed of modal disturbance suppressibility measures, be an $(N \times n)$ matrix defined by

$$H = (\cos\Gamma)V \tag{11}$$

where the matrix $\cos\Gamma$ is composed of each $\cos\gamma_{ik}$. Using these definitions, we may define the gross measure of disturbance suppressibility as follows.

Definition 3.4: The Euclidean norm σ_i of the ith row of H is a gross measure of disturbance suppressibility of the ith mode from all disturbances.

Also, the scalar gross measure of disturbance suppressibility σ_i of the ith mode from all disturbances can be rewritten as follows:

$$\sigma_i = \|h_i\|_2, \qquad h_i = \frac{\psi_i^T E}{\|\psi_i\|_2}, \qquad i = 1, \ldots, N \tag{12}$$

Remarks: Each entry of the vector h_i is the component of a column vector of E in the direction of ψ_i. Therefore, the measure σ_i represents the gross degree of disturbance suppressibility of the ith mode from all disturbances. For systems with repeated eigenvalues, Viswanathan and Longman's[23] or Tarokh's[29] measure of controllability is recommended.

IV. Algorithm for Designing an Effective Controller via Disturbance Accommodating Left Eigenstructure Assignment

In this section, we propose a novel left eigenstructure assignment scheme considering the measure of modal disturbance suppressibility as well as the improved measure of modal controllability. After discussing how to determine the desired left modal matrix Ψ^d, we will propose a new design procedure that assigns eigenvalues to the desired locations exactly and achieves the desired left eigenvectors in the least square sense in the desired eigenvectors do not reside in the achievable subspace.

Determination of a Desired Left Modal Matrix Ψ^d

We consider the problem of defining a family of a desired left modal matrix Ψ^d that has the specified modal controllability and disturbance suppressibility weightings.

Recall the system equation [Eq. (1)] and the relevant assumptions concerning matrices A, B, and E. Each column vector ψ_{ij}^d of the matrix Ψ^d can be generated as follows to reflect the specified modal controllability and disturbance suppressibility weightings:

$$\psi_{ij}^d = \sum_{k=1}^{m} \alpha_k b_k^n + \sum_{l=1}^{\mathrm{rank}[\ker(E)]} \beta_l e_l^{\perp}, \qquad i = 1, \ldots, s \quad j = 1, \ldots, d_i \tag{13}$$

where $0 \leq \alpha_k \leq 1, 0 \leq \beta_l \leq 1, \sum_{k=1}^{m} \alpha_k + \sum_{l=1}^{\mathrm{rank}[\ker(E)]} \beta_l = 1, \alpha_k$ is the controllability weighting corresponding to the kth column of the normalized control input matrix B(i.e., b_k^n), $e_l^{\perp}$ is the lth normalized column of the null space of the disturbance input matrix E, and β_l is the disturbance suppressibility weighting corresponding to $e_l^{\perp}$. Note, for the proposed method of weighting, that the signs of all of the column vectors of Ψ^d, B, and $\ker(E)$ should be adjusted such that the greatest angle between the column vectors is not greater than

90 deg. This prevents two or more column vectors from canceling one another when weightings are added. Now, the matrix Ψ^d can be written as

$$\Psi^d = [\Psi_1^d, \Psi_2^d, \ldots, \Psi_i^d, \ldots, \Psi_s^d], \quad \Psi_i^d = [\psi_{i1}^d, \psi_{i2}^d, \ldots, \psi_{id_i}^d]_{N\times d_i} \tag{14}$$

If the desired eigenvalues are complex, a slight modification of Eq. (14) is required to get better results. Assume that $\lambda_1 = \lambda_2^*$ and all other eigenvalues are real and distinct, which implies $\phi_{11} = \phi_{21}^*$ and $\psi_{11} = \psi_{21}^*$. From these relations, the desired left eigenvectors ψ_{11}^d and ψ_{21}^d, corresponding to the desired complex conjugate eigenvalues λ_1 and λ_2, are reformulated to be complex conjugate to each other as follows:

$$\Psi^d = [\psi_{11}^d + \psi_{21}^d j, \psi_{11}^d - \psi_{21}^d j, \psi_{31}^d, \ldots, \psi_{N1}^d] \tag{15}$$

Best Possible (Achievable) Left Modal Matrix Ψ^a

In this subsection, a new, simple, and general left eigenstructure assignment scheme, avoiding the drawback in the existing method, is derived by introducing the biorthogonality condition between the right and left modal matrices of a system, and the best possible left modal matrix Ψ^a is determined in the least square sense, in case the desired left modal matrix Ψ^d does not reside in the achievable subspace, guaranteeing the desired eigenvalues to be achieved exactly.

For this, we define an $[N \times (N+m)]$ matrix Q_i and an $[(N+m)\times m]$ compatibly partitioned matrix N_i:

$$Q_i \equiv [A - \lambda_i I_N \mid B], \qquad N_i \equiv \begin{bmatrix} N_{1i} \\ -- \\ N_{2i} \end{bmatrix} \tag{16}$$

where the columns of N_i form a basis for the null space of Q_i if the system has distinct eigenvalues. Then, the achievable right eigenvector ϕ_{ij}^a should lie in the span of $\{N_{1i}\}$ for $j = 1, \ldots, d_i$. For rank $B = m$, it can be shown that the columns of N_{1i} are linearly independent.[3] Let us define an achievable generalized right modal matrix Φ^a as follows:

$$\Phi^a = [\Phi_1^a, \Phi_2^a, \ldots, \Phi_i^a, \ldots, \Phi_s^a], \quad \Phi_i^a = [\phi_{i1}^a, \phi_{i2}^a, \ldots, \phi_{id_i}^a]_{N\times d_i} \tag{17}$$

and ϕ_{ij}^a is given as a linear combination of the columns of N_{1i}, that is,

$$\phi_{ij}^a = N_{1i} p_{ij} \tag{18}$$

In Eq. (18), the $(m \times 1)$ coefficient vector p_{ij} is chosen to minimize the following performance index,

$$J = \left\| (\Psi^d)^T \Phi_{\text{aug}}^a P - I_N \right\| \tag{19}$$

where the $(mN \times N)$ coefficient matrix P is formed as follows:

$$P = \text{block diag}[P_1, P_2, \ldots, P_i, \ldots, P_s]$$

with $P_i = \text{block diag}[p_{i1}, p_{i2}, \ldots, p_{id_i}]$. The $(N \times N)$ matrix Ψ^d is determined according to the guideline described in the previous subsection to reflect the specified modal controllability and disturbance suppressibility weightings, and the $(N \times mN)$ augmented achievable right modal matrix Φ_{aug}^a is formed as follows:

$$\Phi_{\text{aug}}^a = [N_{11}, N_{12}, \ldots, N_{1i}, \ldots, N_{1s}] \tag{20}$$

The vector p_{ij} minimizing the performance index J is given by the following equation,

$$p_{ij} = (\Omega_{ij})^\dagger n_k \tag{21}$$

where the superscript $(\cdot)^\dagger$ denotes the pseudo-inverse of a given matrix $(\cdot)$, and the $(N \times m)$ submatrix Ω_{ij} is a component of the following matrix $\{(\Psi^d)^T \Phi_{\text{aug}}^a\}$ of dimension $(N \times mN)$

$$[\Omega_{11}, \Omega_{12}, \ldots, \Omega_{1d_1}, \Omega_{21}, \Omega_{22}, \ldots, \Omega_{2d_2}, \ldots, \Omega_{s1}, \Omega_{s2}, \ldots, \Omega_{sd_s}] = \{(\Psi^d)^T \Phi_{\text{aug}}^a\} \tag{22}$$

and the vector n_k is the kth column of an $(N \times N)$ identity matrix corresponding to the kth submatrix of $\{(\Psi^d)^T \Phi_{\text{aug}}^a\}$. For systems with distinct eigenvalues only, Eqs. (20–22) can be rewritten as follows:

$$\Phi_{\text{aug}}^a = [N_{11}, N_{12}, \ldots, N_{1i}, \ldots, N_{1N}] \tag{23}$$

$$p_{i1} = (\Omega_{i1})^\dagger n_i \tag{24}$$

$$\{(\Psi^d)^T \Phi_{\text{aug}}^a\} = [\Omega_{11}, \Omega_{21}, \ldots, \Omega_{i1}, \ldots, \Omega_{N1}] \tag{25}$$

Then, the achieved right modal matrix Φ^a is given by

$$\Phi^a = \Phi_{\text{aug}}^a P \tag{26}$$

Considering the biorthogonality condition between modal matrices, the achieved generalized left modal matrix Ψ^a, satisfying the design specifications in the least square sense (in case the desired left modal matrix Ψ^d does not reside in the achievable subspace), can be represented by

$$\Psi^a = (\Phi^a)^{-T} \tag{27}$$

Recall that the left eigenstructure assignment scheme by state feedback based upon Theorem 2.1 cannot be directly applied to get the desired left eigenstructure. In our work, however, the drawback may be removed, and the achievable left modal matrix satisfying the prescribed requirements is obtained easily in the least square sense when Ψ^d does not reside in the achievable subspace, assigning exact eigenvalues.

Algorithm

Step 1: Determine the desired eigenvalues and corresponding desired left eigenvectors considering the required modal controllability and disturbance suppressibility weightings according to the guideline [Eq. (13)].

Step 2: Find maximal rank matrices N_i, and S_i for $i = 1, \ldots, s$ such that

$$N_i = \begin{bmatrix} N_{1i} \\ -- \\ N_{2i} \end{bmatrix} \in C^{(N+m)\times m}, \qquad S_i = \begin{bmatrix} S_{1i} \\ -- \\ S_{2i} \end{bmatrix} \in C^{(N+m)\times N}$$

satisfying the relation $[A - \lambda_i I_N \mid B][S_i \mid N_i] = [I_N \mid 0]$.

Step 3: Form the augmented achievable generalized right modal matrix Φ_{aug}^a given by Eq. (20).

Step 4: Select the coefficient vectors $p_{ij}(i = 1, \ldots, s;\ j = 1, \ldots, d_i)$ by Eq. (21) using the desired left modal matrix Ψ^d given in Step 1 and the augmented achievable generalized right modal matrix Φ_{aug}^a given in Step 3.

Step 5: Form the achievable generalized right eigenvectors

$$\phi_{ij}^a = S_{1i}\phi_{ij-1}^a + N_{1i} p_{ij}$$

where $\phi_{i0}^a = 0$, and construct the achievable generalized right modal matrix Φ^a.

Step 6: Calculate the vector chains and construct the matrix W as follows:

$$w_{ij} = S_{2i}\phi_{ij-1}^a + N_{2i} p_{ij},$$

$$W = [w_{11}, w_{12}, \ldots, w_{ij}, \ldots, w_{sd_i}]$$

Step 7: Calculate the state feedback gain matrix

$$K = W(\Phi^a)^{-1}$$

V. Application to a Lateral Flight Control of L-1011 Aircraft

The lateral dynamic model[3] of the L-1011 at a cruise flight condition with disturbances is considered. The model includes actuator dynamics and a washout filter on yaw rate. We design a lateral flight control system for an L-1011 model with wind disturbances, to consider the prescribed modal controllability and disturbance suppressibility weightings, via the proposed left eigenstructure assignment scheme.

The system equations, evaluated at the cruise flight condition, are composed of the following components. The state vector is given by

$$x = [\delta_r, \delta_a, \phi, r, p, \beta, x_7]^T$$

where each state denotes rudder deflection (rad), aileron deflection (rad), bank angle (rad), yaw rate (rad/s), roll rate (rad/s), sideslip angle (rad), and washout filter state, respectively. The A, B, and wind gust disturbances input matrix E are given, respectively, as

$$A = \begin{bmatrix} -20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.744 & -0.032 & 0 & -0.154 & -0.0042 & 1.54 & 0 \\ 0.337 & -1.12 & 0 & 0.249 & -1 & -5.2 & 0 \\ 0.02 & 0 & 0.0386 & -0.996 & -0.000295 & -0.117 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & -0.5 \end{bmatrix}$$

$$B = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \qquad E = \begin{bmatrix} 0 & 0 & 0 & 0.0042 & 1 & 0.000295 & 0 \\ 0 & 0 & 0 & -1.54 & 5.2 & 0.117 & 0 \end{bmatrix}^T$$

The control inputs are

$$u = [\delta_{r_c}, \delta_{a_c}]^T$$

where δ_{r_c} and δ_{a_c} denote rudder command (rad) and aileron command (rad), respectively. We consider the roll and sideslip wind disturbances, i.e., $f = [p_w, \beta_w]^T$. For this system, the set of open-loop eigenvalues is $\Lambda^{\text{open}} = \{\lambda_1^o, \lambda_2^o, \lambda_{3,4}^o, \lambda_5^o, \lambda_6^o, \lambda_7^o\} = \{-20.0, -25.0, -0.0882 \pm 1.2695j, -1.0855, -0.0092, -0.5\}$.

We wish to design a closed-loop controller to provide for the function of a lateral stability augmentation system with closure of the roll attitude loop. As an illustration, we choose the desired closed-loop dutch roll ($\lambda_{3,4}$) and roll ($\lambda_{5,6}$) modes to be $-1.5 \pm 1.5j$ and $-2.0 \pm 1.5j$, respectively, and the other closed-loop eigenvalues are equal to the corresponding open-loop eigenvalues. The null space of E is represented by $E^\perp = [e_1^\perp, e_2^\perp, e_3^\perp, e_4^\perp, e_5^\perp]$. It is assumed that the maximum control surface deflections and deflection rates are not limited in this application to show explicitly the relative performance for the following two cases to be considered.

According to the design procedure of the proposed algorithm in Sec. IV, we determine each left eigenvector, to take account of the prescribed modal controllability and disturbance suppressibility weightings, as follows:

$$\psi_{i1}^d = \sum_{k=1}^{2} \alpha_k^{(i-1)} b_k^n + \sum_{l=1}^{5} \beta_l^{(i-1)} e_l^\perp, \qquad (i = 1, \ldots, 7)$$

where the superscript $(\cdot)^{(i-1)}$ denotes the $(i-1)$th prime notation of $(\cdot)$. The following cases are considered in this application.

Case 1

The modal controllability weightings are accounted more than the following case 2. The specified weightings for modal controllability and disturbance suppressibility are the following:

$$\{\alpha_1, \alpha_1', \alpha_1'', \alpha_1^{(3)}, \alpha_1^{(4)}, \alpha_1^{(5)}, \alpha_1^{(6)}\} = \{0.7, 0.75, 0.65, 0.65, 0.63, 0.63, 0.65\}$$

$$\{\alpha_2, \alpha_2', \alpha_2'', \alpha_2^{(3)}, \alpha_2^{(4)}, \alpha_2^{(5)}, \alpha_2^{(6)}\} = \{0.1, 0.05, 0.15, 0.15, 0.17, 0.17, 0.15\}$$

$$\beta_1^{(i-1)} = \beta_2^{(i-1)} = \beta_3^{(i-1)} = \beta_4^{(i-1)} = \beta_5^{(i-1)} = 0.04, \quad \text{for } i = 1, \ldots, 7$$

After slight modification described in the previous section, since some of the desired eigenvalues ($\lambda_{3,4}$ and $\lambda_{5,6}$) are complex, the desired left modal matrix Ψ^d is obtained as

$$\Psi^d = \begin{bmatrix} 0.6721 & 0.7221 & 0.6221+0.6221j & 0.6221-0.6221j & 0.6021+0.6021j & 0.6021-0.6021j & 0.6221 \\ 0.0532 & 0.0032 & 0.1032+0.1032j & 0.1032-0.1032j & 0.1232+0.1232j & 0.1232-0.1232j & 0.1032 \\ 0.0400 & 0.0400 & 0.0400+0.0400j & 0.0400-0.0400j & 0.0400+0.0400j & 0.0400-0.0400j & 0.0400 \\ 0.0032 & 0.0032 & 0.0032+0.0032j & 0.0032-0.0032j & 0.0032+0.0032j & 0.0032-0.0032j & 0.0032 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0427 & 0.0427 & 0.0427+0.0427j & 0.0427-0.0427j & 0.0427+0.0427j & 0.0427-0.0427j & 0.0427 \\ 0.0400 & 0.0400 & 0.0400+0.0400j & 0.0400-0.0400j & 0.0400+0.0400j & 0.0400-0.0400j & 0.0400 \end{bmatrix}$$

Then, according to the design procedure described in Sec. IV, the normalized achievable left modal matrix Ψ_{nor}^a is given by

$$\Psi_{\text{nor}}^a = \begin{bmatrix} 0.1680 & -0.1661 & -0.0243+0.0163j & -0.0243-0.0163j & 0.0357-0.0260j & 0.0357+0.0260j & 0.0049 \\ 0.0302 & -0.0353 & -0.0028+0.0022j & -0.0028-0.0022j & 0.0038-0.0027j & 0.0038+0.0027j & 0.0005 \\ -0.0861 & 0.1133 & -0.0092-0.0196j & -0.0092+0.0196j & 0.0864-0.0165j & 0.0864+0.0165j & 0.0131 \\ -0.5092 & 0.4753 & 0.3726-0.2753j & 0.3726+0.2753j & -0.4735+0.2264j & -0.4735-0.2264j & -0.0510 \\ -0.0484 & 0.0614 & -0.0011-0.0098j & -0.0011+0.0098j & 0.0169-0.0223j & 0.0169+0.0223j & 0.0026 \\ 0.8371 & -0.8528 & -0.9736-0.1056j & -0.9736+0.1056j & 0.9132+0.1204j & 0.9132-0.1204j & 0.5810 \\ 0.0322 & -0.0361 & -0.0172+0.0015j & -0.0172-0.0015j & 0.0020+0.0026j & 0.0020-0.0026j & 0.8122 \end{bmatrix}$$

and the state feedback gain matrix K is

$$K = \begin{bmatrix} 1.2235 & 0.2953 & -1.3725 & 0.1878 & -0.6864 & 3.3064 & 0.2507 \\ -5.9492 & -1.2080 & 8.3481 & 13.6508 & 4.3853 & -36.0645 & -1.9470 \end{bmatrix}$$

Case 2

The modal disturbance suppressibility weightings are accounted more than the case 1. The specified weightings for modal controllability and disturbance suppressibility are the following:

$$\left\{\alpha_2, \alpha_2', \alpha_2'', \alpha_2^{(3)}, \alpha_2^{(4)}, \alpha_2^{(5)}, \alpha_2^{(6)}\right\} = \{0.45, 0.05, 0.35, 0.35, 0.57, 0.57, 0.35\}$$

$$\left\{\beta_5, \beta_5', \beta_5'', \beta_5^{(3)}, \beta_5^{(4)}, \beta_5^{(5)}, \beta_5^{(6)}\right\} = \{0.35, 0.75, 0.45, 0.45, 0.23, 0.23, 0.45\}$$

$$\alpha_1^{(i-1)} = \beta_1^{(i-1)} = \beta_2^{(i-1)} = \beta_3^{(i-1)} = \beta_4^{(i-1)} = 0.04, \qquad \text{for } i = 1, \ldots, 4, 7$$

$$\alpha_1^{(i-1)} = 0.08, \beta_1^{(i-1)} = \beta_2^{(i-1)} = \beta_3^{(i-1)} = \beta_4^{(i-1)} = 0.03, \qquad \text{for } i = 5, 6$$

In this case, the desired left modal matrix Ψ^d and the normalized achievable left modal matrix Ψ_{nor}^a are given, respectively, in the same way as in case 1, as follows:

$$\Psi^d = \begin{bmatrix} 0.0121 & 0.0121 & 0.0121+0.0121j & 0.0121-0.0121j & 0.0591+0.0591j & 0.0591-0.0591j & 0.0121 \\ 0.4032 & 0.0032 & 0.3032+0.3032j & 0.3032-0.3032j & 0.5349+0.5349j & 0.5349-0.5349j & 0.3032 \\ 0.0400 & 0.0400 & 0.0400+0.0400j & 0.0400-0.0400j & 0.0300+0.0300j & 0.0300-0.0300j & 0.0400 \\ 0.0032 & 0.0032 & 0.0032+0.0032j & 0.0032-0.0032j & 0.0024+0.0024j & 0.0024-0.0024j & 0.0032 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0427 & 0.0427 & 0.0427+0.0427j & 0.0427-0.0427j & 0.0320+0.0320j & 0.0320-0.0320j & 0.0427 \\ 0.3500 & 0.7500 & 0.4500+0.4500j & 0.4500-0.4500j & 0.2300+0.2300j & 0.2300-0.2300j & 0.4500 \end{bmatrix}$$

$$\Psi_{\text{nor}}^a = \begin{bmatrix} 0.0211 & 0.0247 & -0.0023+0.0006j & -0.0023-0.0006j & -0.0021+0.0006j & -0.0021-0.0006j & -0.0004 \\ 0.1796 & 0.1615 & -0.0063-0.0222j & -0.0063+0.0222j & 0.0078-0.0233j & 0.0078+0.0233j & -0.0079 \\ -0.3809 & -0.3820 & 0.5346+0.1426j & 0.5346-0.1426j & 0.5172+0.0759j & 0.5172-0.0759j & 0.1278 \\ -0.3743 & -0.3756 & 0.2235-0.0898j & 0.2235+0.0898j & 0.2267-0.1116j & 0.2267+0.1116j & 0.0494 \\ -0.2822 & -0.2717 & 0.2757-0.0043j & 0.2757+0.0043j & 0.2215-0.0362j & 0.2215+0.0362j & 0.0789 \\ 0.7762 & 0.7827 & -0.7887+0.0699j & -0.7887-0.0699j & -0.8075+0.0071j & -0.8075-0.0071j & 0.3877 \\ -0.0013 & -0.0014 & 0.0029+0.0008j & 0.0029-0.0008j & 0.0030+0.0005j & 0.0030-0.0005j & 0.9081 \end{bmatrix}$$

The state feedback gain matrix K is

$$K = \begin{bmatrix} -1.3042 & -7.3007 & 23.7366 & 24.1383 & 15.3096 & -49.5798 & 0.1036 \\ 0.0785 & 0.8142 & -0.5443 & -0.4298 & -0.1623 & 1.0311 & -0.0041 \end{bmatrix}$$

The Frobenius norms of the feedback gains for the two cases are given by 40.3922 and 62.4170, respectively. It is because, as a design specification, the controllability weightings have been accounted more in case 1, while the disturbance suppressibility weightings have been accounted more in case 2. That is, the desired disturbance suppressibility for case 2 has been achieved in the least square sense at the cost of larger feedback gains than those of case 1. For case 1, the second row of the gain matrix (which corresponds to the aileron deflection command δ_{a_c}) has larger elements than those of the first row (which corresponds to the rudder deflection command δ_{r_c}) in the absolute value sense. On the other hand, the first row of the gain matrix has larger elements than those of the second row for case 2. For these two cases, the desired left eigenvectors are assigned in the least square sense, and the desired eigenvalues are achieved exactly.

The responses of the closed-loop system $\dot{x} = (A + BK)x$ with $\beta(0) = 1$ deg for case 1 and case 2 are shown in Fig. 1 to investigate the regulation performance. The figure shows that both roll rate p and sideslip angle β for case 1 exhibit smaller overshoots and better regulation performances with smaller control efforts compared with those in case 2. This is because case 1 has been designed to have better controllability than case 2.

Now, we assume that wind gust disturbances are present for a sufficiently small time interval. That is, the impulsive disturbances are applied to the system. The impulsive disturbances can be implemented in the simulations by letting the initial condition of the state vector be $x(0) = [0, 0, 0, 1.3535, 0.1, 0.1, 0]^T$. Note that $x(0)$ is determined by the magnitude of the applied disturbances and the disturbance input matrix E.

Figure 2 depicts the suppression performance of the corrupted disturbances in roll rate and sideslip angle for the two cases. All initial conditions are assumed to be zero in these cases. The figure shows that case 2 exhibits slightly better disturbance suppression performance compared with case 1. The results are consistent with our intent in this paper, since case 2 has been designed to have better disturbance suppressibility than case 1.

In the two figures, it can be found that the control surface deflections for the two cases are proportional to the magnitudes of the feedback gains.

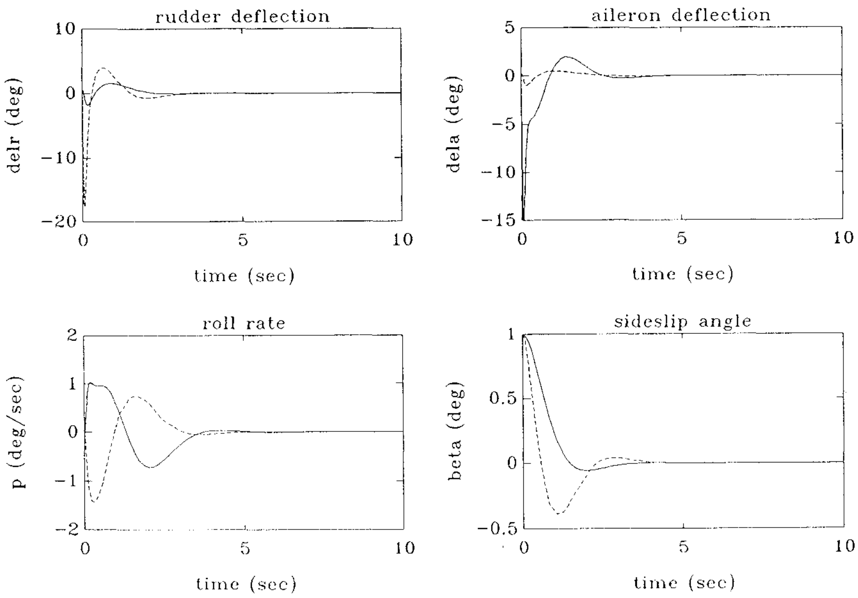

Fig. 1 Closed-loop responses for the two cases with $\beta(0) = 1$ deg (solid line: case 1, and dashed line: case 2).

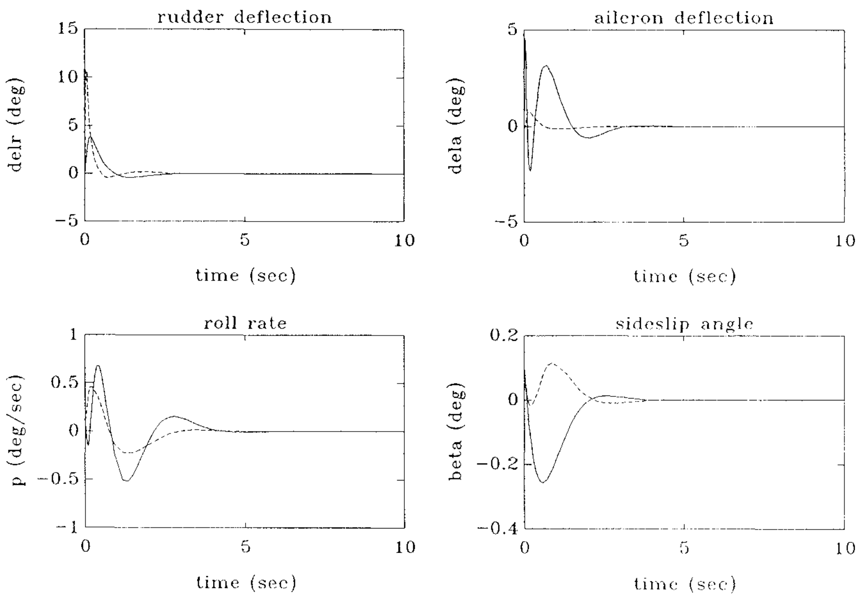

Fig. 2 Disturbed closed-loop responses for the two cases with zero initial conditions (solid line: case 1, and dashed line: case 2).

The resulting output responses shown in the two figures do not show significantly improved performance, since the number of unknown elements (7) of each left eigenvector is greater than the number of independent control actuators [rank$(B) = 2$]. However, if the number of independent control actuators is increased to the number of unknown elements of each left eigenvector, then the desired left eigenstructure is achieved exactly and thus will show significantly improved performance. These facts agree with the results of Srinathkumar.[30]

VI. Concluding Remarks

In this paper, the modal disturbance suppressibility measure and gross disturbance suppressibility measure of a given mode in all disturbances have been proposed. A simple and general left eigenstructure assignment scheme, considering modal disturbance suppressibility as well as the improved modal controllability, has been proposed. The proposed left eigenstructure assignment scheme makes it possible to achieve the desired left eigenstructure exactly if the desired left eigenvectors reside in the achievable subspace. In case the desired left eigenvectors do not reside in the achievable subspace, the left eigenvectors are assigned to the best possible set of eigenvectors in the least square sense, guaranteeing the desired eigenvalues to be achieved exactly. An application to a lateral flight control system design of an L-1011 aircraft model with wind disturbances has demonstrated that the proposed left eigenstructure assignment scheme is also useful in designing control systems considering both control effectiveness and disturbance suppressibility. Finally, it should be noted that the proposed design methodology might be sensitive to parameter variations. The primary objective of this work is to illustrate the proposed left eigenstructure assignment methodology. The robustness problem of the methodology is still an open question for further research.

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